Cosmic Ray Transport with Magnetic Focusing and the “Telegraph” model

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Abstract

Cosmic rays (CR), constrained by scattering on magnetic irregularities, are believed to propagate diffusively. But a well-known defect of diffusive approximation whereby some of the particles propagate unrealistically fast has directed interest towards an alternative CR transport model based on the "telegraph" equation. However, its derivations often lack rigor and transparency leading to inconsistent results.

We apply the classic Chapman-Enskog method to the CR transport problem. We show that no "telegraph" (second order time derivative) term emerges in any order of a proper asymptotic expansion with systematically eliminated short time scales. Nevertheless, this term may formally be *converted* from the *fourth* order hyper-diffusive term of the expansion. But, both the telegraph and hyperdiffusive terms may only be important for a short relaxation period associated with either strong pitch-angle anisotropy or spatial inhomogeneity of the initial CR distribution. Beyond this period the system evolves diffusively in both cases. The term conversion, that makes the telegraph and Chapman-Enskog approaches reasonably equivalent, is possible only after this relaxation period. During this period, the telegraph solution is argued to be unphysical. Unlike the hyperdiffusion correction, it is not uniformly valid and introduces implausible singular components to the solution. These dominate the solution during the relaxation period. As they are shown not to be inherent in the underlying scattering problem, we argue that the telegraph term is involuntarily acquired in an asymptotic reduction of the problem.

1. Preliminary Considerations

The problem addressed here is fundamental and not new to the cosmic ray (CR) transport studies. It can be formulated very plainly: How to describe CR transport by only their isotropic component, after the anisotropic one has decayed by scattering on magnetic irregularities? Suppose the angular distribution of CRs is given by the function $f(\mu, t, z)$ obeying an equation from which the rapid gyro-phase rotation is already removed (drift approximation, e.g., Vedenov et al. 1962; Kulsrud 2005)

$$\frac{\partial f}{\partial t} + v\mu \frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu} \left(1 - \mu^2\right) \mathscr{D}(\mu) \frac{\partial f}{\partial \mu}. \tag{1}$$

Here z is the local coordinate along the ambient magnetic field, μ is the cosine of the particle pitch angle, and $\mathscr{D}$ is the pitch angle diffusion coefficient. Now, we make the next step in simplifying the transport description and seek an equation for the pitch-angle averaged distribution

$$f_0(t,z) \equiv \frac{1}{2}\int_{-1}^{1} f(\mu,t,z)\,d\mu \equiv \langle f \rangle .$$

The basic solution to this problem has been known for at least half a century (e.g., Jokipii 1966 and references therein). To the leading order in $1/\mathscr{D}$ (assuming the characteristic scale and time of the problem being longer than particle mean free path and collision time) it can be obtained straightforwardly by averaging eq.(1)

$$\frac{\partial f_0}{\partial t} = -\frac{v}{2}\frac{\partial}{\partial z}\left\langle \left(1-\mu^2\right)\frac{\partial f}{\partial \mu}\right\rangle ,$$

and substituting $\partial f/\partial \mu \ll f_0$ from eq.(1) as:

$$\frac{\partial f}{\partial \mu} \approx -\frac{v}{2\mathscr{D}}\frac{\partial f_0}{\partial z}. \tag{2}$$

Thus, the following diffusion equation results for f_0:

$$\frac{\partial f_0}{\partial t} = \frac{v^2}{4}\frac{\partial}{\partial z}\left\langle \frac{1-\mu^2}{\mathscr{D}}\right\rangle \frac{\partial f_0}{\partial z}. \tag{3}$$

A questionable point of course is neglecting $\partial f/\partial t$ in favor for $v\partial f/\partial z$ in eq.(2). It is somewhat justified by the small parameter $\mathscr{D}^{-1} \ll 1$ in the *final result*, given by eq.(3), making $\partial f/\partial t$ hopefully small. On the other hand this is true for $\partial f_0/\partial t$ but not necessarily for $\partial f/\partial t$, since the latter may contain also the rapidly decaying anisotropic part $\tilde{f} = f - f_0$ of the initial CR distribution. For $\mathscr{D}t \gtrsim 1$, however, $\tilde{f}$ must die out and neglecting $\partial f/\partial t$ appears plausible for the *long-term* CR transport. At the same time, $\partial f_0/\partial t$ is large when f_0 is very narrow in z initially, such as in the fundamental solution. In the sequel, these aspects of the CR propagation will be a key for choosing an appropriate asymptotic reduction method.

However convincing the justification, the CR diffusion model encounters the problem of a superluminal, or simply “too-fast” particle propagation. Although rather common for diffusive models, the problem is largely ignorable as long as the number of such particles remains small. There are cases, however, such as the propagation of ultra high-energy cosmic rays, where this problem must be addressed (Aloisio et al. 2009). Various attempts, starting as early as in 60s, e.g., (Axford 1965), have been made to devise a better transport equation for CRs. Unfortunately, in our view, they lack mathematical rigor and clarity and sometimes lead to inconsistent results.

In the most recent telegraph model, due to Litvinenko & Schlickeiser (2013), a higher order in $1/\mathscr{D} \ll 1$ term was included by retaining $\partial f/\partial t$, dropped in the simplest derivation above. This

strategy gave rise to an additional $\partial^2 f_0/\partial t^2$ -term in the "master" equation. This additional term transforms eq.(3) into a "telegraph" equation:

$$\frac{\partial f_0}{\partial t}+T\frac{\partial^2 f_0}{\partial t^2}=\frac{\partial}{\partial z}k\frac{\partial f_0}{\partial z}+\frac{k}{L}\frac{\partial f_0}{\partial z} \tag{4}$$

with

$$k=\frac{v^2}{4}\left\langle\frac{1-\mu^2}{\mathscr{D}}\right\rangle,\quad T=\left\langle\left[\int_0^{\mu}d\mu/\mathscr{D}\right]^2\right\rangle\bigg/\left\langle\frac{1-\mu^2}{\mathscr{D}}\right\rangle,\quad L^{-1}=-B^{-1}\partial B/\partial z \tag{5}$$

For the sake of comparison with earlier telegraph equation results that will be made in Sec.4, we have added here the magnetic focusing effect (the last term on the r.h.s. with $B(z)$ being the magnetic field), not included initially in eq.(1). Eq.(4) is just a linear equation that can be solved immediately. The fundamental solution to eq.(4) that starts off from a $\delta(z)$ distribution, instantaneously released at $t=0$, is as follows (e.g., Goldstein 1951; Axford 1965; Schwadron & Gombosi 1994; Litvinenko et al. 2015, $L^{-1}=0$, for simplicity)

$$\begin{aligned} f_0 &= \frac{1}{2}e^{-t/2T}\left\{\delta\left(z-\sqrt{\frac{k}{T}}t\right)+\delta\left(z+\sqrt{\frac{k}{T}}t\right)\right. \\ &+ \left.\frac{H\left(\sqrt{k/T}t-|z|\right)}{2\sqrt{kT}}\left[I_0\left(\frac{1}{2}\sqrt{\frac{t^2}{T^2}-\frac{z^2}{kT}}\right)+\frac{t}{\sqrt{T\left(kt^2-Tz^2\right)}}I_1\left(\frac{1}{2}\sqrt{\frac{t^2}{T^2}-\frac{z^2}{kT}}\right)\right]\right\} \end{aligned} \tag{6}$$

Here $I_{0,1}$ denote the modified Bessel functions and H - the Heaviside unit function.

One promising aspect of the telegraph equation is that it allows for a ballistic mode of CR propagation when the initial conditions empower the higher-order derivative terms to dominate (at least in early phase of evolution). If, in addition, T has a proper value, the bulk speed of CRs may also be realistic. For example, this speed was derived in (Earl 1973) to be $v/\sqrt{3}$, which has also been used earlier by Axford 1965. This is just the rms velocity projection of an isotropic, one-sided CR distribution on z-axis, which appears to resolve the issue with the superluminal propagation. What is worrisome here is that this bulk speed essentially requires a *one-sided* "isotropic" CR distribution which, of course, is highly anisotropic overall, contrary to the basic assumption of most treatments. So, we need to start with isotropic initial distribution but, taking it narrow and symmetric in z (say Gaussian), the higher-order derivative terms will, again, dominate in eq.(4) and the single CR pulse will split into two, propagating in opposite directions at the speeds $\pm\sqrt{k/T}$ (as $\partial f_0/\partial t=0$ at $t=0$, according to eq.[1]). Neglecting $\partial f_0/\partial t$ (which is justified for f_0 sufficiently narrow in z), the solution is simply

$f_0(z,t) = F\left(z - \sqrt{k/T}t\right) + F\left(z + \sqrt{k/T}t\right)$, where $2F(z) = f(z,0)$. This result casts doubts on whether the telegraph term can be dominant under the assumption of frequent CR scattering (asymptotic expansion in small $1/\mathscr{D}$), as the initially sharp profile does not spread. Bringing $\partial f_0/\partial t$ back into the equation will only damp but not spread the profile, as clearly seen from the solution given in eq.(6), where $F\left(z \pm \sqrt{k/T}t\right) = \delta\left(z \pm \sqrt{k/T}t\right)$. Besides, the bulk CR speed $\sqrt{k/T} = v/\sqrt{3}$ for an isotropic scattering $\mathscr{D}$ =1, although implicitly confirmed in the recent derivation of the telegraph equation for an arbitrary $\mathscr{D}(\mu)$ by Litvinenko & Schlickeiser (2013), is not universally accepted. Gombosi et al. (1993); Pauls et al. (1993) and Schwadron & Gombosi (1994), using simplified forms of $D(\mu)$, advocate the value $\sqrt{5/11}v$ for the propagation speed.

The last result is consistent with our calculations below, but with strong reservations regarding the telegraph equation set out later in the paper. Here we merely note that the solution of telegraph equation specifically considered by Litvinenko & Schlickeiser (2013), which does not have the property of splitting the initially narrow pulse into two, does not conserve the total CR number $N = \int f_0 dz$. It starts off from $N = 0$ which is unphysical, as the equation has no source on its r.h.s. To conserve N, two $\delta-$ pulses in eq.(6) are necessary. Those have been added to the treatment by Litvinenko & Noble (2013), but the $\delta-$ pulses have not been shown on their plots, for obvious reason. So, the comparison of this solution with the solution of the original scattering problem is rather misleading. The disagreement on the propagation speed $\sqrt{k/T}$ is also critical as the solution in eq.(6) is cut off at a point z moving with this speed. For $z < \sqrt{k/T}t$ the profile is close to a Gaussian (for $t \gg T$, where T is the scattering time), so small variations in the speed can produce significant variations in the solution. We will also return to this later.

Another disadvantage of the telegraph equation (4) is that it is no longer an evolution equation and requires the time derivative $\partial_t f_0$ as an initial condition. Although this can be inferred from the angular distribution at $t = 0$ using eq.(1), the "telegraph" description of CR transport is not self-contained. We show below that the T-term in eq.(4) is subdominant in an asymptotic series for $\mathscr{D}t \gtrsim 1$, thus representing transients in the CR transport. Strictly speaking, it should be omitted in the asymptotic transport description along with the small hyper-diffusion term $\sim \partial^4 f_0/\partial z^4$, particularly if the term $\sim \partial^3 f_0/\partial z^3$ does not vanish. The latter was not included in eq.(4), as it was obtained by applying insufficient direct iteration to eq.(1) in (Litvinenko & Schlickeiser 2013), or assuming symmetric scattering, $\mathscr{D}(-\mu) = \mathscr{D}(\mu)$, when it indeed vanishes. This symmetry restriction was relaxed in (Pauls et al. 1993).

The reasons why we undertake the derivation of master equation to a higher (fourth) order of approximation for an arbitrary $\mathscr{D}(\mu)$ are several. First, it is necessary to clarify the role of the telegraph term entertained in the literature as an allegedly viable alternative to the standard diffusion model. Second, it is important to obtain the transport coefficients valid for arbitrary $\mathscr{D}(\mu)$, that is for an arbitrary spectrum of magnetic fluctuations. As we will show, the previous such derivation due to Litvinenko & Schlickeiser 2013, does not include the third order term, while including only one fourth order term, while there are more such terms (see eq.[27]). Furthermore, the diffusion equation (3) supplemented by a convective term $u(z)\,\partial f_0/\partial z$ for the

case of the bulk fluid (scattering center) motion with velocity u, has long been and remains the main tool for diffusive shock acceleration (DSA) models. An accurate assessment of the next non-vanishing term, not included in eq.(3), is thus utterly important for the DSA, particularly as claims are being made about the necessity to include the telegraph term in the CR transport. In most DSA applications, it is crucial to allow not only for an arbitrary fluctuation spectrum $\mathscr{D}$ but for its dependence upon f_0 as well. This dependence directly affects the particle spectrum and acceleration time. We will discuss these aspects briefly in Sec.6.

In the next section, the basic transport equation with magnetic focusing is introduced and the shortcomings of a reduction scheme based on direct iterations are demonstrated. The appropriate asymptotic method is elaborated in Sec.3. Apart from what we already discussed regarding the telegraph equation, the objective of Sec.3 is to create a framework suitable also for nonlinear (e.g., Ptuskin et al. 2008; Malkov et al. 2010b) and quasi-linear (Fujita et al. 2011; Malkov et al. 2013) versions of CR transport which are important for both the DSA and for the subsequent escape of the accelerated CR. In these settings, the CR pressure is high enough to strongly modify at least the pitch-angle diffusion coefficient $\mathscr{D}$ and possibly the shock structure itself (Malkov et al. 2010b). In Secs.4 and 5 the implications of our results for the telegraph model and for the long-time CR propagation are discussed, while Sec.6 concludes the paper.

2. CR Transport Equation and its Asymptotic Reduction

Energetic particles (e.g., CRs) in a magnetic field, slowly varying on the particle gyro-scale, are transported according to the following gyro-phase averaged equation, e.g. (Vedenov et al. 1962; Jokipii 1966; Kulsrud 2005)

$$\frac{\partial f}{\partial t}+v\mu\frac{\partial f}{\partial z}+v\frac{\sigma}{2}\left(1-\mu^2\right)\frac{\partial f}{\partial\mu}=\frac{\partial}{\partial\mu}\nu D\left(\mu\right)\left(1-\mu^2\right)\frac{\partial f}{\partial\mu} \tag{7}$$

Here v and μ are the particle velocity and pitch angle, z points in the local field direction, $\sigma=-B^{-1}\partial B/\partial z$ is the magnetic field inverse scale and ν is the pitch angle scattering rate, while $D\left(\mu\right)\sim 1$ depends on the spectrum of magnetic fluctuations. As the fastest transport is assumed to be in μ, we introduce the following small parameter

$$\varepsilon\equiv\frac{v}{l\nu}\equiv\frac{\lambda}{l}\ll 1, \tag{8}$$

where λ is the particle mean free path and l is a characteristic scale that should be chosen depending on the problem considered. One option is the scale of $B(z)$, in which case $l\sim\sigma^{-1}$. If the CR source is present on the r.h.s. of eq.[7]), its scale can be taken as l. Finally, l can be the scale of an initial CR distribution. Strictly speaking, the shortest of these scales should be taken as l. The problem with the initial CR distribution is that in the most interesting case of the fundamental solution this scale is zero. Therefore, over the initial period of CR spreading, before

the actual CR scale $l(t) \sim f/(\partial f/\partial z)$ exceeds the m.f.p. λ, direct asymptotic expansions in small ε remain inaccurate. The goal here is to choose the *least inaccurate* out of all possible expansion schemes. At a minimum, it should be the one that does not introduce additional singularities, apart from the initial delta function $\delta(z)$, that must spread out under the particle recession and collisions. Therefore, while taking $l = const$ in eq.(8), and assuming $l \gg \lambda$, caution will be exercised during the initial phase of the CR relaxation when the terms with higher spatial and time derivatives are large, even if they contain small factors $\varepsilon^n \ll 1$. By measuring time in ν^{-1}, z in l, and simply replacing $\sigma l \to \sigma \sim 1$, the above equation transforms as follows

$$\frac{\partial f}{\partial t} - \frac{\partial}{\partial \mu} D(\mu)\left(1-\mu^2\right)\frac{\partial f}{\partial \mu} = -\varepsilon\left(\mu\frac{\partial f}{\partial z} + \frac{\sigma}{2}\left(1-\mu^2\right)\frac{\partial f}{\partial \mu}\right) \tag{9}$$

A suitable scheme for asymptotic reduction of the above equation using $\varepsilon \ll 1$ is due to Chapman and Enskog, suggested in development of the earlier ideas by Hilbert (a good discussion of the history of this method with mathematical details is given by Cercignani 1988). Originally, it was applied to Boltzmann equation in a strongly collisional regime. Similar approaches have been used in plasma physics, e.g., in regards to the hydrodynamic description of collisional magnetized plasmas (Braginskii 1965) and the problem of run-away electrons (Gurevich 1961; Kruskal & Bernstein 1964).

Regardless of the asymptotic scheme, eq.(9) suggests to seek f as a series in ε

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \ldots \equiv f_0 + \tilde{f} \tag{10}$$

where

$$\langle f \rangle = f_0, \quad \text{with} \quad \langle \cdot \rangle = \frac{1}{2}\int_{-1}^{1} (\cdot)\, d\mu, \tag{11}$$

so that $\langle \tilde{f} \rangle = \langle f_{n>0} \rangle = 0$. The equation for f_0, which is the main ("master") equation of the method, takes the following form

$$\frac{\partial f_0}{\partial t} = -\varepsilon\left(\frac{\partial}{\partial z} + \sigma\right)\langle \mu f \rangle = -\frac{\varepsilon^2}{2}\left(\frac{\partial}{\partial z} + \sigma\right)\sum_{n=1}^{\infty}\varepsilon^{n-1}\left\langle \left(1-\mu^2\right)\frac{\partial f_n}{\partial \mu}\right\rangle \tag{12}$$

We see from this equation that, similarly to the case of Lorentz's gas in an electric field (Gurevich 1961; Kruskal & Bernstein 1964), f_0 depends on the "slow time" $t_2 = \varepsilon^2 t$ rather than on t. Indeed, the two problems are similar in that they describe diffusive expansion of particles in phase space. The expansion occurs in z-direction for the CR diffusion problem and in energy for runaway electrons. The expansion is driven by a rapid isotropization in pitch angle plus the convection in z-

direction, or acceleration in the electric field direction, for the CR transport and electron runaway, respectively.

The slow dependence of f_0 on time in eq.(12) may suggest to attribute the time derivative term in eq.(9) to a higher order approximation (thus moving it to the r.h.s.). Such ordering has been employed by Litvinenko & Schlickeiser (2013) and the term $\propto \partial^2 f_0/\partial t^2$ has been produced in eq.(12). Obviously, a continuation of this process would result in progressively higher time derivatives of f_0, corresponding to shorter and shorter times in the initial relaxation. These transient phenomena will be removed using the Chapman-Enskog asymptotic reduction scheme in the next section.

Unlike f_0, $\tilde{f}$ in eq.(10) does depend on t as on a "fast" time. Therefore, it is illegitimate to attribute the first term on the l.h.s of eq.(9) to any order of approximation different from that of the second term, notwithstanding its fast decay for $t \gtrsim 1$. Thus, using eqs.(9-10) we must apply the following ordering

$$\frac{\partial f_n}{\partial t} - \frac{\partial}{\partial \mu} D(\mu)\left(1-\mu^2\right)\frac{\partial f_n}{\partial \mu} = -\mu\frac{\partial f_{n-1}}{\partial z} - \frac{\sigma}{2}\left(1-\mu^2\right)\frac{\partial f_{n-1}}{\partial \mu} \tag{13}$$

The above expansion scheme is sufficient to recover the leading order of f_0 evolution from eq.(12) by substituting there $\partial f_1/\partial\mu \approx -(2D)^{-1}\partial f_0/\partial z$, obtained from the last equation for $t \gtrsim 1$. However, this scheme is not suitable for determining f_n for $n \geq 2$ to submit to eq.(12). Indeed, as it may be seen from eq.(13), the solubility condition for f_2 at $t \gg 1$ is $(\partial/\partial z + \sigma)\left\langle\left(1-\mu^2\right)\partial f_1/\partial\mu\right\rangle \approx -(\partial/\partial z+\sigma)\left\langle\left(1-\mu^2\right)/2D\right\rangle\partial f_0/\partial z = 0$. This is clearly too strong a restriction. The reason for this inconsistency of the direct asymptotic expansion is that f_0 depends on time much slower than $f_{n>0}$, so a slow time $t_2 = \varepsilon^2 t$ needs to be taken into consideration. The Chapman-Enskog method has been developed for such cases, and we will make use of it in the next section.

3. Chapman-Enskog Expansion

As we have seen, the asymptotic reduction of the original CR propagation problem, given by eq.(9), to its isotropic part cannot proceed to higher orders of approximation using a simple asymptotic series in eq.(10) and requires a multi-time asymptotic expansion. In Chapman-Enskog method the operator $\partial/\partial t$ is expanded instead. Its purpose is to avoid unwanted higher time derivatives to appear in higher orders of approximation. This is very similar to, e.g., a secular growth in perturbed oscillations of dynamical systems. To eliminate the secular terms, one seeks to alter (also expand in small parameter) the frequency of the zero order motion, which is similar to the $\partial/\partial t$ expansion. One example of such approach may be found in a derivation of hydrodynamic equations for strongly collisional but magnetized plasmas, starting from Boltzmann equation (Mikhailovsky 1967). The classical monograph by Chapman & Cowling (1991) (Ch.VIII) gives another example of a subdivision of $\partial/\partial t$ operator for solving the transport

problem in a non-uniform gas-mixture. Expanding $\partial/\partial t$ operators eliminates secular terms, such as the telegraph term. Perhaps more customary today and equivalently is to introduce a hierarchy of formally independent time variables (e.g., Nayfeh 1981) $t \to t_0,\ t_1, \ldots$, so that

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1} + \varepsilon^2 \frac{\partial}{\partial t_2} \ldots \tag{14}$$

Instead of eq.(13), from eq.(9) we have

$$\begin{aligned} \frac{\partial f_n}{\partial t_0} - \frac{\partial}{\partial \mu} D(\mu)\left(1-\mu^2\right)\frac{\partial f_n}{\partial \mu} &= -\mu \frac{\partial f_{n-1}}{\partial z} - \frac{\sigma}{2}\left(1-\mu^2\right)\frac{\partial f_{n-1}}{\partial \mu} - \sum_{k=1}^{n} \frac{\partial f_{n-k}}{\partial t_k} \\ &\equiv \mathscr{L}_{n-1}[f](t_0, \ldots, t_n; \mu, z) \end{aligned} \tag{15}$$

where the conditions $f_{n<0} = 0$ are implied. The solution of this equation should be sought in the following form

$$f_n = \overline{f_n}(t_2, t_3, \ldots; \mu) + \tilde{f}_n(t_0, t_1, \ldots; \mu) \tag{16}$$

where $\tilde{f}_n$ and $\overline{f}_n$ are chosen such to satisfy, respectively, the following two equations:

$$\frac{\partial \tilde{f}_n}{\partial t_0} - \frac{\partial}{\partial \mu} D(\mu)\left(1-\mu^2\right)\frac{\partial \tilde{f}_n}{\partial \mu} = \mathscr{L}_{n-1}\left[\tilde{f}\right](t_0, \ldots, t_n; \mu, z) \tag{17}$$

and

$$-\frac{\partial}{\partial \mu} D(\mu)\left(1-\mu^2\right)\frac{\partial \overline{f_n}}{\partial \mu} = \mathscr{L}_{n-1}\left[\overline{f}\right](t_2, \ldots, t_n; \mu, z) \tag{18}$$

The solution for $\tilde{f}_n$ is as follows

$$\tilde{f}_n = \sum_{k=1}^{\infty} C_k^{(n)}(t_0)\, e^{-\lambda_k t_0} \psi_k(\mu) \tag{19}$$

and it can be evaluated for arbitrary n by expanding both sides of eq.(17) in a series of eigenfunctions of the diffusion operator on its l.h.s.:

$$-\frac{\partial}{\partial \mu} D(\mu)\left(1-\mu^2\right)\frac{\partial \psi_k}{\partial \mu} = \lambda_k \psi_k,$$

For $D = 1$, for example, ψ_k are the Legendre polynomials with $\lambda_k = k(k+1)$, $k = 0, 1, \ldots$. The time dependent coefficients $C_k^{(n)}$ are determined by the initial values of $\tilde{f}_n$ (anisotropic part of the initial CR distribution) and the r.h.s. of eq.(17), that depends on $\tilde{f}_{n-1}$, obtained at the preceding

step. It is seen, however, that $\tilde{f}_n$ exponentially decay in time for $t \gtrsim 1$ and we may ignore them[1] as we are primarily interested in evolving the system over times $t \gtrsim \varepsilon^{-2} \gg 1$ and even longer. Starting from $n = 0$ and using eq.(15), for the slowly varying part of f we have

$$\frac{\partial f_0}{\partial t_0} = 0. \tag{20}$$

The solubility condition for f_1 (obtained by integrating both sides of eq.[15] in μ) also gives a trivial result

$$\frac{\partial f_0}{\partial t_1} = 0, \tag{21}$$

so the last two conditions are consistent with the suggested decomposition in eq.(16), since from eq.(18) with $n = 1$ we have

$$\bar{f_1} = -\frac{1}{2} W \frac{\partial f_0}{\partial z} \tag{22}$$

and, thus both $\bar{f_0}$ and $\bar{f_1}$ are, indeed, independent of t_0 and t_1. We have introduced the function $W(\mu)$ here by the following two relations

$$\frac{\partial W}{\partial \mu} = \frac{1}{D}, \quad \langle W \rangle = 0. \tag{23}$$

The solubility condition for f_2 yields the nontrivial and well-known (e.g., Jokipii 1966) result, which is actually the leading term of the $\partial f_0 / \partial t$ expansion in $\varepsilon \ll 1$

$$\frac{\partial f_0}{\partial t_2} = \frac{1}{4} \left(\frac{\partial}{\partial z} + \sigma \right) \kappa \frac{\partial f_0}{\partial z}, \tag{24}$$

where

$$\kappa = \left\langle \frac{(1 - \mu^2)}{D} \right\rangle .$$

The solubility conditions for f_3, $f_4, \ldots$ will generate the higher order terms of our expansion which, after some algebra, can be manipulated into the following expressions for the third and fourth orders of approximation

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[1]In fact we must do so, as our asymptotic method has a power accuracy in $\varepsilon \ll 1$, but not the exponential accuracy.

$$
\begin{aligned}
\frac{\partial f_0}{\partial t_3} &= -\frac{1}{4}\left(\frac{\partial}{\partial z}+\sigma\right)\left(\frac{\partial}{\partial z}+\frac{\sigma}{2}\right)\left\langle \mu W^2\right\rangle \frac{\partial f_0}{\partial z} && (25)\\
\frac{\partial f_0}{\partial t_4} &= \frac{1}{8}\left(\frac{\partial}{\partial z}+\sigma\right)\times \\
& \left\{\left(\frac{\partial}{\partial z}+\frac{\sigma}{2}\right)^2\left\langle W^2\left(U'-\kappa\right)\right\rangle+\frac{1}{2}\left(\frac{\partial}{\partial z}+\sigma\right)\frac{\partial}{\partial z}\left\langle\frac{\left[\kappa\left(1-\mu\right)+U\right]^2}{D\left(1-\mu^2\right)}\right\rangle\right\}\frac{\partial f_0}{\partial z}. && (26)
\end{aligned}
$$

We have denoted

$$
U \equiv \int_1^\mu \frac{1-\mu^2}{D} d\mu,
$$

and $U' = \partial U/\partial\mu$. The pitch-angle diffusion coefficient $D(\mu)$ and magnetic focusing σ are considered z- independent for simplicity, a limitation that can be easily relaxed by re-arranging the operators containing $\partial/\partial z$ in eq.(26). We can proceed to higher orders of approximation *ad infinitum* since terms containing $\left\langle\left(1-\mu^2\right)\partial f_n/\partial\mu\right\rangle$ can be expressed through $f_{n-1},\ f_{n-2},\ldots$. According to eqs.(20-21), of interest is the evolution of f_0 on the time scales $t_2 \gtrsim 1$ or $t \gtrsim \varepsilon^{-2}$ so, as we already mentioned, the contributions of $\tilde{f}_n(\mu)$ to all the solubility conditions, similar to those given by eqs.(24-26), have to be dropped (as they become exponentially small) and only $\overline{f_n}(\mu)$- contributions should be retained. Using eqs.(20-21,24-26) to form the combinations $\varepsilon^n\partial^n f_0/\partial t_n$ and summing up both sides, on the l.h.s. of the resulting equation we simply obtain $\partial f_0/\partial t$ (see eq.[14]). Therefore, the evolution of f_0 up to the fourth order in ε takes the following form

$$
\frac{\partial f_0}{\partial t} = \frac{\varepsilon^2}{4}\partial_z'\left\{\kappa-\varepsilon\partial_z''\left\langle\mu W^2\right\rangle-\frac{\varepsilon^2}{2}\left[K_1\left(\partial_z''\right)^2-K_2\partial_z'\partial_z\right]\right\}\frac{\partial f_0}{\partial z} \tag{27}
$$

where $\partial_z' = \partial_z+\sigma$, $\partial_z'' = \partial_z+\sigma/2$, and

$$
K_1 = \left\langle W^2\left(\kappa-U'\right)\right\rangle,\quad K_2 = \frac{1}{2}\left\langle\frac{\left[\kappa\left(1-\mu\right)+U\right]^2}{D\left(1-\mu^2\right)}\right\rangle \tag{28}
$$

The above algorithm allows one to obtain the master equation to arbitrary order in ε. By construction, in no order of approximation will higher time derivatives emerge, as has been devised by Chapman and Enskog. We have truncated this process at the fourth order, ε^4. As we show in the next section, this is the lowest order required to relate the above result to the telegraph equation. It also gives the first non-vanishing correction to the standard CR diffusion model in an important case $\left\langle\mu W^2\right\rangle = 0$, which is fulfilled, in particular, for $D(-\mu) = D(\mu)$. Higher order terms can be calculated at the expense of a more involved algebra, but we argue below that such calculations would not change the results significantly.

4. Comparison with Earlier Results. Recovering Telegraph Term

In contrast to the telegraph equation given by eqs.(4-5), that has been derived by Litvinenko & Schlickeiser (2013) using direct iteration of eq.(7) with no explicit ordering of the emerging terms, eq.(27) is derived to the ε^4- order of approximation with an ε^n factor labeling each term. Yet, it has no second order time derivative which is inconsistent with eq.(4). Below we demonstrate that eq.(27) can still be converted to the telegraph form, however, with additional terms absent from eq.(4-5). Although eq.(27) is obtained by a broadly a applicable Chapman-Enskog method, its reduction to the telegraph form below is more restrictive and should be taken with a grain of salt for the reasons we discuss later.

Several versions of telegraph equation have been obtained using different methods but, unfortunately, many of them do not offer clear ordering, as e.g., eq.[4] derived by Litvinenko & Schlickeiser 2013. In an earlier treatment by (Earl 1973), an eigenfunction expansion was truncated with no transparent assessment of discarded terms. As we mentioned already, many treatments do not systematically eliminate short time scales which are irrelevant to the long-time evolution of the isotropic part of the CR distribution. In principle, this is acceptable if the reduction scheme is based on an exact solution of the original equation, to include all required orders of approximation into the master equation. Such approach, along with a nearly exhaustive analysis of the previous work has been presented in (Schwadron & Gombosi 1994). Their treatment, however, is by necessity limited to a relatively simple $D(\mu)$ (i.e., power-law in μ).

To clarify the role of the higher order terms in eq.(27), we note that the r.h.s. of this equation represents just the first three non-vanishing contributions from an infinite asymptotic series (in $\varepsilon \ll 1$) which we would obtain by continuing the reduction process described in the preceding section. This series may or may not converge to some linear (integral) operator in z. From a practical standpoint, the maximum order term that needs to be retained is either the first non-vanishing term, or else it introduces a new property to the solution, such as symmetry breaking. Precisely the last aspect has been highlighted by Litvinenko & Schlickeiser (2013) who used the telegraph term to calculate the skewness of a CR pulse. From this angle, we examine the third and the fourth order below separately.

Third order equation. To this order eq.(27) rewrites

$$\frac{\partial f_0}{\partial \tau} = V \frac{\partial f_0}{\partial z} + \kappa_1 \frac{\partial^2 f_0}{\partial z^2} - \varepsilon \left\langle \mu W^2 \right\rangle \frac{\partial^3 f_0}{\partial z^3}. \tag{29}$$

We have introduced the slow time $\tau = \varepsilon^2 t/4$, as a natural time scale for the reduced system, and the following notation

$$V = \sigma \left(\kappa - \frac{1}{2} \varepsilon \sigma \left\langle \mu W^2 \right\rangle \right), \quad \kappa_1 = \kappa - \frac{3}{2} \varepsilon \sigma \left\langle \mu W^2 \right\rangle.$$

Note, that using τ instead of t makes the terms looking lower by two orders in ε, but describing them in the sequel we will use their original order, as it stands in eq.(27) rather than eqs.(30) or (32).

Eq.(29) can be solved using a Fourier transform and integral representations of Airy functions. We consider some basic properties of this solution using the moments of f_0. In particular, it is seen from this equation that the skewness of a CR pulse, propagating at the bulk speed V, arises in this order of approximation. Indeed, upon a Galilean transform to the reference frame moving with the speed $-V$, $z \to z' = z + V\tau$, the above equation rewrites

$$\frac{\partial f_0}{\partial \tau} = \kappa_1 \frac{\partial^2 f_0}{\partial z'^2} - \varepsilon \left\langle \mu W^2 \right\rangle \frac{\partial^3 f_0}{\partial z'^3} \tag{30}$$

so the last term generates an antisymmetric component of $f_0(z')$, even if $f_0(z')$ is an even function of z' initially. By normalizing f_0 to unity

$$\overline{f_0} = \int_{-\infty}^{\infty} f_0 dz' = 1,$$

and assuming the coefficients in eq.(30) to be constant, for the moments of $f_0(z', t)$

$$\overline{z'^n} = \int_{-\infty}^{\infty} f_0 z'^n dz',$$

we obtain

$$\frac{d}{d\tau}\overline{z'} = 0, \quad \frac{d}{d\tau}\overline{z'^2} = 2\kappa_1, \quad \frac{d}{d\tau}\overline{z'^3} = 6\kappa_1 \overline{z'} + 6\varepsilon \left\langle \mu W^2 \right\rangle.$$

With no loss of generality we may set $\overline{z'} = 0$ and, in addition, $\overline{z'^3} = 0$ at $\tau = 0$, so that the skewness of the CR distribution changes in time as follows

$$S \equiv \frac{\overline{z'^3}}{\left(\overline{z'^2}\right)^{3/2}} = \frac{6\varepsilon \left\langle \mu W^2 \right\rangle \tau}{\left(\overline{z_0'^2} + 2\kappa_1 \tau\right)^{3/2}}$$

where '0' at $\overline{z'^2}$ refers to its value at $\tau = 0$. The skewness remains small and its maximum

$$S_{\max} = \frac{2\varepsilon \left\langle \mu W^2 \right\rangle}{\kappa_1 \sqrt{3\overline{z_0'^2}}}$$

is achieved at $\tau = \overline{z'^2}/\kappa_1$. Not surprisingly, the skewness increases with decreasing initial spatial dispersion of CRs. Indeed, according to eq.(9), a narrow $f(z)$ generates strong pitch-angle anisotropy which, in combination with asymmetric pitch-angle scattering ($\langle \mu W^2 \rangle \neq 0$), generates the spatial skewness of the CR pulse. For not too small τ, the explicit form of the solution of eq.(30) may be easily written down by using, e.g., a Fourier transform in z' and the steepest descent estimate of its inversion

$$f\left(z',\tau\right) \simeq \frac{C}{\sqrt{\tau}} \exp\left[-\frac{z'^2}{4\kappa_1 \tau}\left(1-\frac{\varepsilon z'}{2\kappa_1^2 \tau}\left\langle \mu W^2 \right\rangle\right)\right]. \tag{31}$$

It is quite possible, however, that even this small effect does not occur because of the pitch angle scattering symmetry, that is $\langle \mu W^2 \rangle = 0$. In this case the solution remains diffusive and, to obtain corrections to it and to see where the telegraph term might come from, the next approximation needs to be considered.

Fourth order equation. The telegraph term. In the absence of ε^3 terms, that is when $\langle \mu W^2 \rangle = 0$, eq.(27) takes the following form

$$\frac{\partial f_0}{\partial \tau} + \varepsilon^2 \left(K_1 - K_2\right)\left(\sigma + \frac{1}{2}\frac{\partial}{\partial z}\right)\frac{\partial^3 f_0}{\partial z^3} = \kappa_2 \frac{\partial^2 f_0}{\partial z^2} + V_2 \frac{\partial f_0}{\partial z} + \mathscr{O}\left(\varepsilon^4\right) \tag{32}$$

$$\kappa_2 = \kappa - \frac{\sigma^2 \varepsilon^2}{2}\left(\frac{5}{4}K_1 - K_2\right), \quad V_2 = \sigma\left[\kappa - \frac{\sigma^2\varepsilon^2}{8}K_1\right]$$

This equation, obtained within Chapman-Enskog method, and the telegraph equation (4), obtained by a direct iteration method, differ from each other in the second term on the l.h.s. The remaining terms of the two equations are equivalent even though not identical due to the insignificant $\sim \varepsilon^4$ corrections included in the coefficients κ_2 and V on the r.h.s. of eq.(32).

To understand how the conflicting terms on the l.h.s. of both equations are related, we note that within the regular ordering scheme leading to eq.(32) the term in question must remain small, being nominally an ε^4-term. However, as the conflicting terms in both equations are the higher-order derivatives, they may stick out from their order of approximation if the solution strongly varies in space and time. In order to preserve the overall solution integrity in such events a multi- (time)-scale or matched asymptotic expansion method is normally applied. We will argue that the telegraph equation approach to the CR transport does not handle this situation properly, as opposed to the Chapman-Enskog approach. But this is not to say that the two terms cannot be mapped to each other, when they are well in the validity range in the above sense. Note that in a number of other treatments the telegraph term was tacitly handled as one of the dominant terms. We pointed out that indeed, the term in question on the l.h.s of eq.(32) is small only insofar as the z- derivative does not change its order of approximation due to strong inhomogeneity, whose

scale should not be less than the m.f.p., λ. So, assuming strong inequality $\partial_z \ll \lambda^{-1}$ (or simply $\partial_z \sim 1$ in our dimensionless variables), we can express the high order spatial derivatives using a zero order ($\varepsilon \to 0$) version of this equation with sufficient accuracy. The result reads:

$$\frac{\partial f_0}{\partial \tau} + \frac{\varepsilon^2}{2\kappa^2}(K_1 - K_2)\frac{\partial^2 f_0}{\partial \tau^2} = \left(\kappa - \frac{\sigma^2 \varepsilon^2}{8} K_1\right)\frac{\partial^2 f_0}{\partial z^2} + V_2 \frac{\partial f_0}{\partial z} + \mathscr{O}\left(\varepsilon^4\right), \tag{33}$$

This equation is indeed equivalent to the telegraph equation by its form, but the equivalence requires not only $\varepsilon \ll 1$ but also smooth variation in z and τ, as not to raise the actual value of these terms significantly. Under these equivalence conditions, both the telegraph and the hyperdiffusive transport terms are just the corrections and may be safely ignored (especially if the ε^3 contribution is not empty). On the contrary, when the higher derivatives strongly enhance these terms, the equations cannot be mapped to each other and their solutions are disparate. One of them (or even both) may become less accurate than the underlying leading order (diffusive) approximation. This is quite common situation in asymptotic expansions when the form of the next order term should be selected on the ground of the least possible singularity it introduces into the expansion (cf. small denominator, secular growth etc. in mechanical problems, where the higher order approximations, if handled blithely, only aggravate disagreement with true solutions). The telegraph term correction to the diffusive approximation appears to come from that variety, as it generates singular components (δ- and Heaviside functions) that are not only inconsistent with the strong pitch angle scattering and resulting spatial diffusion, but with the scatter-free limit of the parent differential equation itself. Therefore, the singular part of the telegraph solution is inherited from the derivation of telegraph equation. We compare the telegraph and hyperdiffusion type corrections to the basic diffusive propagation somewhat further in the next section.

5. Relation between Telegraph and Hyperdiffusion Approximation

We start with a relatively minor aspect of the differences between the two models. As we stated in Sec.1, the telegraph coefficient in eq.(33) is inconsistent with some of the earlier derivations. In the simplest case $D = 1$, for example, after proper rescaling of τ and z, it turns out to be smaller than the term T in eq.(4) by a factor $11/15$. On the other hand, this is consistent with the respective result obtained by Gombosi et al. (1993); Pauls et al. (1993) and Schwadron & Gombosi (1994). While the above difference may be considered rather quantitative, in the general case of $D(-\mu) \neq D(\mu)$, the appropriate equation for describing CR transport is that given by the lower, ε^3- order, not included in eq.(4).

More importantly, the telegraph version of eq.(32) given by eq.(33) is valid only if the telegraph term ($\propto \varepsilon^2$, fourth order term) remains small compared to the other terms and the original ordering in eq.(32) is not violated by strong variations of the solution in space and time, as we pointed out earlier. We signify this by the “slow” time $\tau \sim \varepsilon^2 t$. In most other treatments t is used instead, which formally makes the telegraph term in eq.(33) appearing as a zero order

term. It is not important, of course, whether the term is labeled by ε^2 or not; important is that it is treated as a subordinate term. Attempts to make it dominant *a posteriori* violates assumptions that are essential for its derivation. This point is demonstrated below by repeating a simple calculation of the CR pulse skewness that we already made earlier working to ε^3 order.

Litvinenko & Schlickeiser (2013) suggested to study an asymmetry (skewness) of a CR pulse propagating along the field under the action of magnetic focusing using the telegraph equation. This and some other characteristics of the CR pulse, such as the kurtosis, can be easily analyzed using the primary equation (32). The calculation of the pulse skewness essentially repeats the one already done at the ε^3 level, where it is generated by asymmetric scattering, $D(-\mu) \neq D(\mu)$. So, transforming eq.(32) to the reference frame moving with the speed $-V_2$, that is $z \to z' = z + V_2\tau$, we obtain

$$\frac{d}{d\tau}\overline{z'} = 0, \quad \frac{d}{d\tau}\overline{z'^2} = 2\kappa_2, \quad \frac{d}{d\tau}\overline{z'^3} = 6\varepsilon^2\sigma\left(K_1 - K_2\right), \tag{34}$$

where we have, again, assumed $\overline{z'} = 0$. The skewness thus evolves in time as follows

$$S = \frac{6\varepsilon^2\sigma\left(K_1 - K_2\right)\tau}{\left(\overline{z_0'^2} + 2\kappa_2\tau\right)^{3/2}}$$

Unless $S(\tau)$ reaches its maximum very early it is fairly small. Because $\tau_{\max} = \overline{z_0'^2}/\kappa_2$, the maximum value

$$S_{\max} = S\left(\tau_{\max}\right) = \frac{2\varepsilon^2\sigma\left(K_1 - K_2\right)}{\kappa_2\sqrt{3\overline{z_0'^2}}}, \tag{35}$$

so that an initially symmetric CR pulse develops significant asymmetry only if $\overline{z_0'^2} \lesssim \varepsilon^4$. This, however, would require $\tau_{\max} \sim \varepsilon^4$ (or, equivalently, $t_{\max} \sim \varepsilon^2$), in strong violation of the requirement $t \gtrsim 1$, established in Sec.3. Note that significant pulse asymmetry obtained by Litvinenko & Schlickeiser (2013) using the telegraph equation was based on the fundamental solution to this equation, that is $z_0' \to 0$. We argued in Sec.1 that the early propagation phase is not adequately described by the telegraph equation so that the pulse asymmetry might have been overestimated in the above paper. We specify the validity range of the telegraph equation below.

Starting from a long time regime $\tau > \varepsilon z'$, similarly to the ε^3 result given in eq.(31), from eq.(32) we find

$$f\left(z', \tau\right) \simeq \frac{C}{\sqrt{\tau}}\exp\left\{-\frac{z'^2}{4\kappa_2\tau}\left[1 - \varepsilon^2\frac{\left(K_1 - K_2\right)z'}{2\kappa_2\tau}\left(\sigma - \frac{z'}{4\kappa_2\tau}\right)\right]\right\} \tag{36}$$

This propagation regime is not much different from the regular diffusion (first term in the square bracket), so both the hyperdiffusion and telegraph models produce similar results, as they are largely equivalent in this regime. It is worthwhile to write the requirement for the agreement between the two models in physical units which is simply

$$vt > z'$$

The point $z = vt$ is close to the cut-off in the telegraph solution, $z/t = \sqrt{k/T}$ (for $\sigma = 0$), eq.(6). It follows then that unless $z^2 \gg kt$ ($t \gg T$), the cut-off strongly changes the overall solution.

The opposite case $\tau < \varepsilon z'$, which corresponds to the initial phase of pulse relaxation, is the key to understanding the difference between the Chapman-Enskog and the telegraph methods. In this regime they are *not equivalent*, as the ε^4 term in eq.(32) cannot be neglected to make the transition to the telegraph equation (33). Indeed, when propagation starts with an infinitely narrow pulse, in the early phase of its relaxation the higher z- derivatives are still too large for such transformation. A spatially narrow pulse automatically generates strong pitch angle anisotropy which, in turn, results in rapid time variation, making the telegraph term also large. It is this regime where both methods become questionable and, in addition, their predictions deviate from each other both *quantitatively and qualitatively*. We need to check first whether they are relevant to this regime.

The phase $\tau < \varepsilon z'$ is well described by the Chapman-Enskog approach down to $\tau \sim 1/\varepsilon^2$ ($t_0 \sim 1$, Sec.3). The situation with the telegraph equation is more complex, as the conversion from the Chapman-Enskog expansion is invalid, while independent derivations rarely provide clear ordering. A rigorous derivation in (Schwadron & Gombosi 1994) requires the same assumptions that we made when transforming the hyperdiffusive equation into the telegraph equation, that is $\partial_\tau \sim \partial_z^2$ ($\varepsilon_\tau \sim \varepsilon_\lambda^2$ under their nomenclature). So, the telegraph equation appears to be a subset of the hyperdiffusion equation valid only under the above ordering. It is likely to break down in the $\tau < \varepsilon z'$ regime, in other words, near its cut-off. We support this premise by the following considerations.

Recently, Effenberger & Litvinenko (2014) and Litvinenko et al. (2015) have carried out simulations of the full scattering problem, corresponding to eq.(9). The results deviate from the telegraph solution precisely at the early phase of the pulse propagation, when the hyperdiffusion and telegraph models disagree. The two δ- function pulses with sharp fronts in its solution given by eq.(6) are not seen in the simulations. This is understandable, as such features are inconsistent with the underlying scattering problem. They should have been smeared out by scattering earlier, since the spatial profile is shown at five collision times (Fig.2 in Effenberger & Litvinenko 2014). Moreover, the $\delta-$ function pulses $\delta\left(z \pm \sqrt{k/T}t\right)$ that are an integral part of the telegraph solution, as they maintain its normalization, are irrelevant to the primary equation (1), even without collisions. Indeed, if $\mathscr{D} = 0$, and the initial condition is $\delta(z)$ being constant in μ for $-1 < \mu < 1$, the scatter-free solution is $\delta(z - \mu vt)$. Hence, $f_0(z,t) \equiv \int f d\mu/2 = (2vt)^{-1} H(vt - |z|)$. Therefore, a certain property of the telegraph equation allows the $\delta\left(z \pm \sqrt{k/T}t\right)$ and sharp

front components (although with modified speed, eq.[6]) to survive multiple collisions. As these components are inconsistent with the underlying scattering problem, this property of the equation must have been acquired during its derivation. Obviously, it is rooted in the hyperbolic (telegraph) operator $\partial_t^2 - (k/T)\,\partial_z^2$ that allows the singular profiles to propagate without spreading.

By contrast, the hyperdiffusion equation does not require singular components but, on the contrary, smears them out. We present an approximate solution of eq.(32) after neglecting magnetic focusing in ε^4- order terms in eq.(32). We also neglect the regular diffusion compared to the hyperdiffusion, which is acceptable during an early phase of pulse relaxation, $\tau < \varepsilon z'$. Finally, we assume $z' > 0$, as the solution is an even function of z'. The asymptotic result is as follows:

$$f = \frac{2}{3}\sqrt{\frac{2}{\pi}}\,(4h\tau)^{-1/6}\,z'^{-1/3}\exp\left(-\frac{3}{8}4^{-1/3}\frac{z'^{4/3}}{h^{1/3}\tau^{1/3}}\right)\cos\left(\frac{3^{3/2}}{8}4^{-1/3}\frac{z'^{4/3}}{h^{1/3}\tau^{1/3}}\right). \tag{37}$$

We have denoted the hyperdiffusion constant $h = \varepsilon^2\,(K_1 - K_2)\,/2$. More about this result and further discussion of the two conflicting approaches can be found in Appendix. We see that there is a considerable slow down of the CR spreading compared to the conventional diffusion, $z'^2 \propto t^{1/2}$, that embodies a sub-diffusive propagation, $z'^2 \propto t^{1/4}$. This ameliorates the problem of acasual propagation in diffusion regime, yet no sharp fronts or spikes develop. By contrast, the telegraph solution to the causality problem is to cut off the solution beyond certain distance ($|z| > \sqrt{k/T}t$), thus introducing an unphysical singularity. The immediately arising normalization problem is then "solved" by adding an even stronger singularity in form of two δ functions at the cut-off points.

6. Summary and Conclusions

Using the Chapman-Enskog method, we have extended the CR transport equation with magnetic focusing to the fourth order in a small parameter $\varepsilon = \lambda/l$ (CR mean free path to the characteristic scale of the problem). This analysis clarifies the nature of the telegraph transport equation, widely publicized in the literature as a promising alternative to diffusive propagation models. We have shown that the telegraph extension ($\propto \partial^2 f_0/\partial t^2$) of the diffusion equation can be mapped from the (small) hyper-diffusive term ($\propto \partial^4 f_0/\partial z^4$) of the regular Chapman-Enskog expansion, but the telegraph term, originating from an ε^4 term of the expansion, must remain subordinate to the main, diffusive transport and magnetic focusing (if present) contributions. This condition is met after $\nu t > z/\lambda$ collision times for an initially narrow ($\Delta z < \lambda$) CR distribution. Another important limitation of the telegraph equation is that, by contrast to the Chapman-Enskog equation and the original pitch-angle scattering equation, it is not self-contained requiring an initial condition for also $\partial f_0/\partial t$, which needs information about the anisotropic part of the initial CR distribution. Furthermore, an attempt to proceed to higher orders in ε introduces progressively shorter time scales associated with "ghost" terms reflecting quick relaxation of initial anisotropy

or strong spatial inhomogeneity. By contrast, the classic Chapman-Enskog method is devised to eliminate short time scales, irrelevant to the evolution of the isotropic part of CR distribution f_0, which accurately describes this evolution after a few collision times, $\nu t > 1$.

We have derived the CR transport equation for an arbitrary pitch-angle scattering coefficient $D(\mu)$. This form of transport equation, (27), is suitable for describing CR acceleration and escape problems where the phenomenon of self-confinement (D is a functional of f, $D = D[f;\mu,t]$) is critical, e.g. (Ptuskin et al. 2008; Malkov et al. 2010b, 2013; Fujita et al. 2011). Accounting for magnetic focusing effects is required, e.g., for describing particle acceleration in CR-modified shocks with an oblique magnetic field. In this case the field increases towards the shocks due to the pressure exerted by the accelerated CRs on the flow, thus producing a mirror effect. The particle drift velocity along the field associated with the mirror effect is (e.g., eq.[29]) $V \sim \kappa/l_B$ which, for the magnetic field variation scale being of the order of the shock precursor scale $\kappa/U_{\rm sh}$ and strong shock modification, almost automatically becomes comparable with the shock velocity $U_{\rm sh}$. This additional bulk motion of the accelerated CR (directed towards the shock) will affect their spectrum and acceleration time.

Furthermore, as the CR scattering in such environments (i.e., supernova remnant [SNR] shocks) must be self-sustained by virtue of instabilities of the CR distribution (see, e.g., Bykov et al. (2013); Bell (2014) for the recent reviews), the above magnetic drift needs to be included in the CR stability analysis. It should be noted, however, that the results obtained in the present paper formally require a magnetic field B_0 that does not strongly change over the gyro-radius of energetic particles. This is not to be expected in SNR shocks, especially if strong, CR current- and pressure-driven instabilities generate fields with $\delta B > B_0$. However, one may use the shock normal direction as the polar axis to calculate the pitch angle diffusion coefficient $D(\mu)$, needed for the description of the CR spatial transport. Also, such treatment will require a description of the gyro-motion and averaging by computing particle orbits beyond the standard quasi-linear description (Malkov & Diamond 2006), implied throughout this paper.

In conclusion, by comparison with the telegraph equation, the classic Chapman-Enskog hyper-diffusion equation consistently describes the long-term CR propagation in a self-contained, order controlled fashion. Further improvement of the CR diffusion models should probably address the anisotropic component of the CR distribution. There are situations, such as ultra-high energy CR propagation, where the mean free path grows too long with the energy as to make the diffusive approach irrelevant and a rectilinear transport to dominate (Aloisio et al. 2009) (cf. Levi flight regime described in aforementioned study by Malkov & Diamond 2006). Another interesting example is a sharp angular anisotropy $\sim 10^{\circ}$ in CR arrival directions discovered by MILAGRO observatory (Abdo et al. 2008) and a number of other instruments, e.g. (Abbasi et al. 2011; Bartoli et al. 2013; Desiati 2014; Abeysekara et al. 2014). Unless this anisotropy is of a very local origin (such as heliosphere, (Lazarian & Desiati 2010; O'C. Drury 2013), it poses a real challenge to CR propagation models and clearly cannot be addressed within the diffusive approaches discussed in this paper (Drury & Aharonian 2008; Malkov et al. 2010a; Giacinti & Sigl 2012; Ahlers 2014; Malkov 2015). On the other hand, when the diffusive transport

model is well within its validity range (weakly anisotropic spatially smooth CR distributions) neither the telegraph nor the hyper-diffusive term (both$\sim \varepsilon^4$) is essential to the CR transport and can be neglected.

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A. Appendix

To derive the result given by eq.(37), we rewrite eq.(32) for a simple case of weak focusing ($\sigma f < \partial_z f$) and using the Galilean transform to the frame moving at the magnetic drift velocity V_2, $z' = z + V_2 \tau$:

$$\frac{\partial f}{\partial \tau} = \kappa_2 \frac{\partial^2 f}{\partial z'^2} - h \frac{\partial^4 f}{\partial z'^4}, \tag{A1}$$

where $h = \varepsilon^2 \left(K_1 - K_2\right)/2$ and κ_2 is the same as in eq.(32). The fundamental solution of eq.(A1) can be written as an inversion of the Fourier image $f_k\left(\tau\right)$, assuming that $f_k\left(0\right) = 1/2\pi$. As we are going to find the Fourier inversion using asymptotic methods, we write an arbitrary constant C instead of this value:

$$f\left(\tau, z'\right) = C \int_{-\infty}^{\infty} e^{ikz' - k^2\left(\kappa_2 + hk^2\right)\tau} dk. \tag{A2}$$

We will determine the normalization constant C shortly from the requirement $f \to \delta\left(z'\right),\ \tau \to 0$. It is convenient to introduce the following notation:

$$\zeta^4 = 4h\tau k^4,\ \ \xi = z' \left(4h\tau\right)^{-1/4} > 0.$$

We may limit our consideration to the $\xi \geq 0$ half-space, because the solution is an even function of z'. Focusing on a short time asymptotic regime $\tau < \varepsilon z'$, which is opposite to the case considered earlier (eq.[36]) we neglect the diffusive term $\sim k^2$ in the exponent (as hyperdiffusion dominates) and rewrite eq.(A2) as follows

$$f\left(\tau, z'\right) = \frac{C}{\left(4h\tau\right)^{1/4}} \int_{-\infty}^{\infty} e^{i\xi\zeta - \zeta^4/4} d\zeta \tag{A3}$$

Note that the general evaluation of the integral in eq.(A2), with the diffusive term included, is not difficult but more cumbersome. The phase of the integral here has three saddle points, and the following two should be on the integration path

$$\zeta_{\pm} = i\xi^{1/3} e^{\pm i\pi/3},$$

since the integrand reaches its maxima at these points. So, the integration runs from $-\infty$ through ζ_+ to $+i\infty$ then to ζ_- and, finally to $+\infty$. The contributions from the two saddle points then yield

$$f = \frac{2C\sqrt{\pi/3}}{(h\tau)^{1/4}\,\xi^{1/3}} \exp\left(-\frac{3}{8}\xi^{4/3}\right) \cos\left(\frac{3^{3/2}}{8}\xi^{4/3}\right) \tag{A4}$$

The normalization $\int f dz' = 1$ requires the constant $C = 1/\sqrt{3}\pi$, which only insignificantly deviates from the exact value $C = 1/2\pi$ that, in turn, follows from the integral representation of f in eq.(A2) for $\tau \to 0$. Note that we could have replaced an oscillating exponential tail of this solution by zero beyond the point $\xi > (4\pi)^{3/4} 3^{-9/8}$. Such modification of the hyperdiffusive solution would be in the spirit of the telegraph cut-off at $z/t = \sqrt{k/T}$, however, with an essential difference of being only a discontinuity in the solution derivative. The unphysical (oscillatory) behavior beyond the first zero point of the solution in eq.(A4) results from neglecting the diffusion term in eq.(A1), asymptotic methods used to calculate the integral in eq.(A3), and from lacking higher order terms, $\sim \varepsilon^n$, $n > 4$. Therefore, the solution can be improved systematically. In addition, it starts from a point source which is clearly inconsistent with the main approximation $\varepsilon \ll 1$. A somewhat broader initial profile will not develop an oscillatory tail, if convolved with the Green's function in eq.(A2). We will not attempt to improve on this minor aspect of the solution here, as it becomes only weakly irregular if cut off at its first zero.

From the perspective of a general improvement of the asymptotic expansion considered in this paper, the derivation of eqs.(34), for example, is robust in the following sense. The residual higher order terms in $\varepsilon \ll 1$, if included in eq.(32), will not change eqs.(34) in any other way than small corrections to the coefficients κ_2 and V_2. Indeed, the higher z- derivatives coming from higher order terms, will vanish from the (first four) moment equations after integrating by parts. By contrast, continuing the telegraph approach to higher orders will generate terms with small parameters at higher time derivatives in all moment equations. These terms will become crucial during the initial relaxation of the CR distribution. The relaxation is associated with the CR anisotropy or strong initial inhomogeneity, that is with large $\tilde{f}_n$, Sec.3. However, these decay over a short time $t \lesssim 1$. This is the time period when the telegraph or hyperdiffusive correction is large but its effect on the subsequent evolution ought to be limited as this time is short. The hyperdiffusive correction meets this requirement, as we argued using moment equations. To see whether the same is true for the telegraph correction, let us rewrite eq.(33) using the "fast" time, $t = \tau/\varepsilon^2$:

$$\left(1+\tau_{\mathrm{T}}\frac{\partial}{\partial t}\right)\frac{\partial f_0}{\partial t}=\frac{\varepsilon^2}{4}\kappa\frac{\partial^2 f_0}{\partial z^2}, \tag{A5}$$

where we denoted $\tau_{\mathrm{T}}=2\left(K_1-K_2\right)/\kappa\sim1$ and assumed $\sigma=0$, to make the following simple argument. Namely, in the limit $\varepsilon\to0$ there are two modes, of which the first is $f_0=f_0\left(z\right)$. This is the main diffusion mode that slowly evolves in time when $0<\varepsilon\ll1$, and, as we are interested in the evolution over the time scales $t\gtrsim\varepsilon^{-2}$, the telegraph term becomes $\sim\varepsilon^4$ and can be discarded. The second mode corresponds to a rapid decay of the initial distribution $\sim\exp\left(-t/\tau_{\mathrm{T}}\right)$ which is associated with the decay of initial anisotropy or strong inhomogeneity. If this mode is active ($\partial f_0/\partial t\neq0$ in eq.[A5]), then even the total number of particles N is not conserved automatically. So, turning to the moments of eq.(A5) we need to impose the initial condition, $\partial N/\partial t=0$, to ensure the particle conservation. This probably means that $\varepsilon\to0$ is a difficult limit for the telegraph reduction scheme. The singular components in the telegraph solution (6) appear to be primarily associated with the particle conservation problem. The initial relaxation phase ($t\lesssim1$) perhaps, cannot be adequately described by the telegraph reduction scheme using an equation for f_0 alone, as it does not properly “average out” an anisotropic component $\tilde{f}$, which is large during this period of time. The telegraph term is therefore to be understood as a “ghost” term reflecting rapid decay of such components. It follows that the rapidly changing part $\tilde{f}$ in the decomposition in eq.(16) needs to be retained in the short-time analysis along with f_0. Otherwise, the telegraph operator generates unphysical δ- pulses and sharp fronts, just to conserve the number of particles, as discussed earlier. These considerations are, however, not nearly complete. Further useful analysis of propagation modes in the context of telegraph equation can be found in (Schwadron & Gombosi 1994).

To conclude this Appendix we make yet another argument in disfavor of the telegraph equation that is partially related to the above considerations. A consistent asymptotic reduction method must be continuable to infinity in powers of small ε. The Chapman-Enskog scheme clearly is. The outcome will be a series of terms $\sim\partial_z^n f_0$ on the r.h.s. of eq.(27) with just $\partial_t f_0$ on its l.h.s. To solve the resulting equation, only the initial distribution $f_0\left(0,z\right)$ is needed, as the equation remains *evolutionary* and (generalized) parabolic, as its pitch-angle diffusion superset is. The telegraph equation, on the contrary, turns hyperbolic and non-evolutionary after the reduction from the superset equation. By continuing the telegraph reduction scheme to higher orders of approximation, progressively higher *time derivatives* will emerge (along with higher space derivatives). The resulting equations will thus be non-evolutionary and a growing set of initial time derivatives $\partial^n f_0$ will then be needed to solve the initial value problem. These data can be extracted only from the full anisotropic distribution with recourse to the full (anisotropic) equation. Therefore, the telegraph equation is not self-contained and cannot be improved systematically. Any attempts to improve it will introduce shorter and shorter time scales that would require a return to the full anisotropic description.

REFERENCES

Abbasi, R., Abdou, Y., Abu-Zayyad, T., Adams, J., Aguilar, J. A., Ahlers, M., Altmann, D., Andeen, K., Auffenberg, J., Bai, X., Baker, M., Barwick, S. W., Bay, R., Alba, J. L. B., Beattie, K., Beatty, J. J., Bechet, S., Becker, J. K., Becker, K. H., Benabderrahmane, M. L., BenZvi, S., Berdermann, J., Berghaus, P., Berley, D., Bernardini, E., Bertrand, D., Besson, D. Z., Bindig, D., Bissok, M., Blaufuss, E., Blumenthal, J., Boersma, D. J., Bohm, C., Bose, D., Boeser, S., Botner, O., Brown, A. M., Buitink, S., Caballero-Mora, K. S., Carson, M., Chirkin, D., Christy, B., Clem, J., Clevermann, F., Cohen, S., Colnard, C., Cowen, D. F., D'Agostino, M. V., Danninger, M., Daughhetee, J., Davis, J. C., De Clercq, C., Demiroers, L., Denger, T., Depaepe, O., Descamps, F., Desiati, P., de Vries-Uiterweerd, G., DeYoung, T., Diaz-Velez, J. C., Dierckxsens, M., Dreyer, J., Dumm, J. P., Ehrlich, R., Eisch, J., Ellsworth, R. W., Engdegard, O., Euler, S., Evenson, P. A., Fadiran, O., Fazely, A. R., Fedynitch, A., Feintzeig, J., Feusels, T., Filimonov, K., Finley, C., Fischer-Wasels, T., Foerster, M. M., Fox, B. D., Franckowiak, A., Franke, R., Gaisser, T. K., Gallagher, J., Gerhardt, L., Gladstone, L., Gluesenkamp, T., Goldschmidt, A., Goodman, J. A., Gora, D., Grant, D., Griesel, T., Gross, A., Grullon, S., Gurtner, M., Ha, C., Hajismail, A., Hallgren, A., Halzen, F., Han, K., Hanson, K., Heinen, D., Helbing, K., Herquet, P., Hickford, S., Hill, G. C., Hoffman, K. D., Homeier, A., Hoshina, K., Hubert, D., Huelsnitz, W., Huelss, J. P., Hulth, P. O., Hultqvist, K., Hussain, S., Ishihara, A., Jacobsen, J., Japaridze, G. S., Johansson, H., Joseph, J. M., Kampert, K. H., Kappes, A., Karg, T., Karle, A., Kenny, P., Kiryluk, J., Kislat, F., Klein, S. R., Koehne, J. H., Kohnen, G., Kolanoski, H., Koepke, L., Kopper, S., Koskinen, D. J., Kowalski, M., Kowarik, T., Krasberg, M., Krings, T., Kroll, G., Kurahashi, N., Kuwabara, T., Labare, M., Lafebre, S., Laihem, K., Landsman, H., Larson, M. J., Lauer, R., Luenemann, J., Madajczyk, B., Madsen, J., Majumdar, P., Marotta, A., Maruyama, R., Mase, K., Matis, H. S., Meagher, K., Merck, M., Meszaros, P., Meures, T., Middell, E., Milke, N., Miller, J., Montaruli, T., Morse, R., Movit, S. M., Nahnhauer, R., Nam, J. W., Naumann, U., Niessen, P., Nygren, D. R., Odrowski, S., Olivas, A., Olivo, M., O'Murchadha, A., Ono, M., Panknin, S., Paul, L., de los Heros, C. P., Petrovic, J., Piegsa, A., Pieloth, D., Porrata, R., Posselt, J., Price, C. C., Price, P. B., Przybylski, G. T., Rawlins, K., Redl, P., Resconi, E., Rhode, W., Ribordy, M., Rizzo, A., Rodrigues, J. P., Roth, P., Rothmaier, F., Rott, C., Ruhe, T., Rutledge, D., Ruzybayev, B., Ryckbosch, D., Sander, H. G., Santander, M., Sarkar, S., Schatto, K., Schmidt, T., Schoenwald, A., Schukraft, A., Schultes, A., Schulz, O., Schunck, M., Seckel, D., Semburg, B., Seo, S. H., Sestayo, Y., Seunarine, S., Silvestri, A., Slipak, A., Spiczak, G. M., Spiering, C., Stamatikos, M., Stanev, T., Stephens, G., Stezelberger, T., Stokstad, R. G., Stoessl, A., Stoyanov, S., Strahler, E. A., Straszheim, T., Stuer, M., Sullivan, G. W., Swillens, Q., Taavola, H., Taboada, I., Tamburro, A., Tepe, A., Ter-Antonyan, S., Tilav, S., Toale, P. A., Toscano, S., Tosi, D., Turcan, D., van Eijndhoven, N., Vandenbroucke, J., Van Overloop, A., van Santen, J., Vehring, M., Voge, M., Walck, C., Waldenmaier, T., Wallraff, M., Walter, M., Weaver, C., Wendt, C., Westerhoff, S., Whitehorn, N., Wiebe, K., Wiebusch, C. H., Williams, D. R., Wischnewski, R., Wissing, H., Wolf, M., Wood,

T. R., Woschnagg, K., Xu, C., Xu, X. W., Yodh, G., Yoshida, S., Zarzhitsky, P., Zoll, M., & Collaboration, I. 2011, ASTROPHYSICAL JOURNAL, 740

Abdo, A. A., Allen, B., Aune, T., Berley, D., Blaufuss, E., Casanova, S., Chen, C., Dingus, B. L., Ellsworth, R. W., Fleysher, L., Fleysher, R., Gonzalez, M. M., Goodman, J. A., Hoffman, C. M., Hüntemeyer, P. H., Kolterman, B. E., Lansdell, C. P., Linnemann, J. T., McEnery, J. E., Mincer, A. I., Nemethy, P., Noyes, D., Pretz, J., Ryan, J. M., Parkinson, P. M. S., Shoup, A., Sinnis, G., Smith, A. J., Sullivan, G. W., Vasileiou, V., Walker, G. P., Williams, D. A., & Yodh, G. B. 2008, Physical Review Letters, 101, 221101

Abeysekara, A. U., Alfaro, R., Alvarez, C., Álvarez, J. D., Arceo, R., Arteaga-Velázquez, J. C., Ayala Solares, H. A., Barber, A. S., Baughman, B. M., Bautista-Elivar, N., Belmont, E., BenZvi, S. Y., Berley, D., Bonilla Rosales, M., Braun, J., Caballero-Mora, K. S., Carramiñana, A., Castillo, M., Cotti, U., Cotzomi, J., de la Fuente, E., De León, C., DeYoung, T., Diaz Hernandez, R., Díaz-Vélez, J. C., Dingus, B. L., DuVernois, M. A., Ellsworth, R. W., Fiorino, D. W., Fraija, N., Galindo, A., Garfias, F., González, M. M., Goodman, J. A., Gussert, M., Hampel-Arias, Z., Harding, J. P., Hüntemeyer, P., Hui, C. M., Imran, A., Iriarte, A., Karn, P., Kieda, D., Kunde, G. J., Lara, A., Lauer, R. J., Lee, W. H., Lennarz, D., León Vargas, H., Linnemann, J. T., Longo, M., Luna-García, R., Malone, K., Marinelli, A., Marinelli, S. S., Martinez, H., Martinez, O., Martínez-Castro, J., Matthews, J. A. J., McEnery, J., Mendoza Torres, E., Miranda-Romagnoli, P., Moreno, E., Mostafá, M., Nellen, L., Newbold, M., Noriega-Papaqui, R., Oceguera-Becerra, T., Patricelli, B., Pelayo, R., Pérez-Pérez, E. G., Pretz, J., Rivière, C., Rosa-González, D., Ruiz-Velasco, E., Ryan, J., Salazar, H., Salesa Greus, F., Sandoval, A., Schneider, M., Sinnis, G., Smith, A. J., Sparks Woodle, K., Springer, R. W., Taboada, I., Toale, P. A., Tollefson, K., Torres, I., Ukwatta, T. N., Villaseñor, L., Weisgarber, T., Westerhoff, S., Wisher, I. G., Wood, J., Yodh, G. B., Younk, P. W., Zaborov, D., Zepeda, A., Zhou, H., & The HAWC Collaboration. 2014, *Astrophys. J.* , 796, 108

Ahlers, M. 2014, Physical Review Letters, 112, 021101

Aloisio, R., Berezinsky, V., & Gazizov, A. 2009, *Astrophys. J.* , 693, 1275

Axford, W. I. 1965, *Plan. Space Sci.* , 13, 1301

Bartoli, B., Bernardini, P., Bi, X. J., Bolognino, I., Branchini, P., Budano, A., Calabrese Melcarne, A. K., Camarri, P., Cao, Z., Cardarelli, R., Catalanotti, S., Chen, S. Z., Chen, T. L., Creti, P., Cui, S. W., Dai, B. Z., D'Amone, A., Danzengluobu, De Mitri, I., D'Ettorre Piazzoli, B., Di Girolamo, T., Di Sciascio, G., Feng, C. F., Feng, Z., Feng, Z., Gou, Q. B., Guo, Y. Q., He, H. H., Hu, H., Hu, H., Iacovacci, M., Iuppa, R., Jia, H. Y., Labaciren, Li, H. J., Liguori, G., Liu, C., Liu, J., Liu, M. Y., Lu, H., Ma, X. H., Mancarella, G., Mari, S. M., Marsella, G., Martello, D., Mastroianni, S., Montini, P., Ning, C. C., Panareo, M., Panico, B., Perrone, L., Pistilli, P., Ruggieri, F., Salvini, P., Santonico, R., Sbano, S. N., Shen, P. R., Sheng, X. D., Shi, F., Surdo, A., Tan, Y. H., Vallania, P., Vernetto, S., Vigorito,

C., Wang, H., Wu, C. Y., Wu, H. R., Xue, L., Yan, Y. X., Yang, Q. Y., Yang, X. C., Yao, Z. G., Yuan, A. F., Zha, M., Zhang, H. M., Zhang, L., Zhang, X. Y., Zhang, Y., Zhaxiciren, Zhaxisangzhu, Zhou, X. X., Zhu, F. R., Zhu, Q. Q., & Zizzi, G. 2013, Phys. Rev. D, 88, 082001

Bell, A. R. 2014, Brazilian Journal of Physics

Braginskii, S. I. 1965, Transport Processes in a Plasma, Vol. 1, 205

Bykov, A. M., Brandenburg, A., Malkov, M. A., & Osipov, S. M. 2013, *Space Sci. Rev.* , 178, 201

Cercignani, C. 1988, BoltzmannEquation, Applied Mathematical Sciences Series (Springer-Verlag)

Chapman, S., & Cowling, T. G. 1991, The Mathematical Theory of Non-uniform Gases

Desiati, P. 2014, Nuclear Instruments and Methods in Physics Research A, 742, 199

Drury, L. O. C., & Aharonian, F. A. 2008, Astroparticle Physics, 29, 420

Earl, J. A. 1973, *Astrophys. J.* , 180, 227

Effenberger, F., & Litvinenko, Y. E. 2014, *Astrophys. J.* , 783, 15

Fujita, Y., Takahara, F., Ohira, Y., & Iwasaki, K. 2011, *Mon. Not. R. Astron. Soc.* , 415, 3434

Giacinti, G., & Sigl, G. 2012, Physical Review Letters, 109, 071101

Goldstein, S. 1951, The Quarterly Journal of Mechanics and Applied Mathematics, 4, 129

Gombosi, T. I., Jokipii, J. R., Kota, J., Lorencz, K., & Williams, L. L. 1993, *Astrophys. J.* , 403, 377

Gurevich, A. V. 1961, Soviet Journal of Experimental and Theoretical Physics, 12, 904

Jokipii, J. R. 1966, *Astrophys. J.* , 146, 480

Kruskal, M. D., & Bernstein, I. B. 1964, Physics of Fluids, 7, 407

Kulsrud, R. M. 2005, Plasma physics for astrophysics

Lazarian, A., & Desiati, P. 2010, *Astrophys. J.* , 722, 188

Litvinenko, Y. E., Effenberger, F., & Schlickeiser, R. 2015, ArXiv e-prints

Litvinenko, Y. E., & Noble, P. L. 2013, *Astrophys. J.* , 765, 31

Litvinenko, Y. E., & Schlickeiser, R. 2013, Astronomy and Astrophys., 554, A59

Malkov, M. A. 2015, ArXiv e-prints

Malkov, M. A., & Diamond, P. H. 2006, *Astrophys. J.* , 642, 244

Malkov, M. A., Diamond, P. H., O'C. Drury, L., & Sagdeev, R. Z. 2010a, *Astrophys. J.* , 721, 750

Malkov, M. A., Diamond, P. H., & Sagdeev, R. Z. 2010b, Plasma Physics and Controlled Fusion, 52, 124006

Malkov, M. A., Diamond, P. H., Sagdeev, R. Z., Aharonian, F. A., & Moskalenko, I. V. 2013, *Astrophys. J.* , 768, 73

Mikhailovsky, A. B. 1967, Soviet Journal of Experimental and Theoretical Physics, 25, 623

Nayfeh, A. H. 1981, Introduction to perturbation techniques

O'C. Drury, L. 2013, ArXiv e-prints

Pauls, H. L., Burger, R. A., & Bieber, J. W. 1993, International Cosmic Ray Conference, 3, 183

Ptuskin, V. S., Zirakashvili, V. N., & Plesser, A. A. 2008, Advances in Space Research, 42, 486

Schwadron, N. A., & Gombosi, T. I. 1994, *J. Geophys. Res.* , 99, 19301

Vedenov, A. A., Velikhov, E. P., & Sagdeev, R. Z. 1962, NUCLEAR FUSION, 465
